Conic-Based Multiple View Geometry

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Abstract

This paper presents a study, based on conic correspondences, on the relationship between multiple images acquired by uncalibrated cameras. Representing image conics as points in the five-dimensional projective space allows us to handle image conics in the same way as image points. We show that the coordinates of corresponding image conics satisfy the multilinear constraints, as shown in the case for points and lines. To be more specific, the coordinates of two corresponding image conics satisfy bilinear constraints. When a third image comes in, the coordinates of three corresponding image conics satisfy trilinear constraints. Moreover, these constraints are naturally extended to the case where more images are available.

1. Introduction

The appearance of an object's shape varies significantly with changes in viewpoint and this results in numerous different images even for the same object. One of the fundamental difficulties in recognizing objects from images is how to deal with such images obtained from the same object. Clarifying the relationship between images of the same object is thus of fundamental importance in computer vision. Knowledge of the relationship between images provides us with many advantages in important problems in computer vision and multimedia including three-dimensional reconstruction from multiple images, object recognition, image synthesis and image coding.

It is widely known that the coordinates of corresponding points in two perspective images satisfy the bilinear (epipolar) constraint. With the help of projective geometry, it was shown that the coordinates of corresponding points in three uncalibrated perspective images satisfy the trilinear constraints [4, 5, 23, 24, 25, 29]. Faugeras–Mourrain [4] and Triggs [29] independently extended this result to the case of four images, showing that the coordinates of corresponding points in four uncalibrated perspective images satisfy the quadrilinear constraints. Trilinear constraints and quadrilinear constraints have been deeply investigated to understand them in a common framework (see [9, 10, 12, 13, 14], for example) and are called, as a whole, the *multilinear constraints*. These multilinear constraints have been also extended to be applicable to the time-continuous case or the space-continuous case [1, 11, 30].

In the case of lines, by contrast, it is widely known that no constraint exists on the coordinates of two corresponding image lines. When a third image in involved, however, the coordinates of three corresponding image lines satisfy the trilinear constraints [4, 6, 7]. In particular, Hartley [6, 7, 8] showed that the trilinear constraints for lines are essentially equivalent to those for points and opened a unified linear approach to handling both points and lines.

A conic is one of the most important image features. This is because many man-made objects have circular parts, and circles are perspectively projected onto conics. Furthermore, the conic is a more compact primitive than points or lines and can be more robustly and more exactly extracted from images. In addition, finding correspondences between conics is much easier than that between points. Unlike points, conics have features that distinguish one from another and can be used to narrow down the possible matches. It is thus clear that investigating vision problems based on conic correspondence is also significant. Nevertheless, there are fewer articles (for instance, [15, 16, 17, 18, 19, 20, 31, 32]) dealing with conics, and the relationship between corresponding image conics is still an open problem.

This paper presents a study, based on conic correspondences, on the relationship between multiple images acquired by uncalibrated cameras. We employ the representation where a conic is represented as a point in the five-dimensional projective space. This representation allows us to handle image conics in the same way as image points. We show that the coordinates of corresponding image conics satisfy the multilinear constraints, as shown in the case for points and lines. In particular, we focus on the two-image

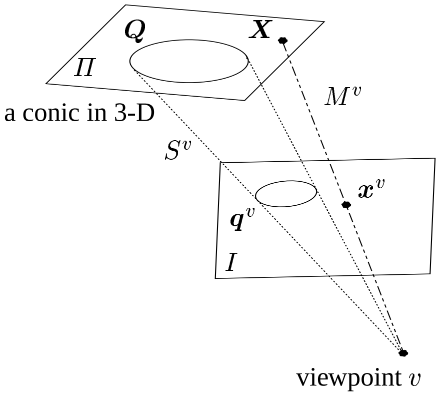

Figure 1. Pointwise transformation and conicwise transformation.

case and the three-image case. We show that the coordinates of two corresponding image conics satisfy five independent bilinear constraints. When a third image comes in, the coordinates of three corresponding image conics satisfy trilinear constraints. Moreover, these constraints are naturally extended to the case where more images are available. Note that a part of this work was presented in [27] and [28].

2. Pointwise projection

We discuss here the relationship between coplanar points in 3-D and their images based on the framework of projective geometry. Note that if not explicitly stated, the coordinates are understood to be homogeneous throughout this paper. An introduction to elementary projective geometry can be found in [3] or [22].

Let $\mathcal{P}^n$ be the n-dimensional projective space over the real number field $\mathbf{R}$. When we observe points on plane Π in 3-D, we embed Π in 3-D into $\mathcal{P}^2$. We also embed the image plane I (v is the viewpoint) in $\mathcal{P}^2$. Embedding Π and I in $\mathcal{P}^2$ allows us to express the transformation from Π to I as a plane projective transformation. That is, letting the homogeneous coordinates of a point in Π in 3-D and its image observed from viewpoint v be $\boldsymbol{X}$ and $\boldsymbol{x}^v$ respectively, we have

$$\boldsymbol{x}^v \simeq (M^v)^{-1}\boldsymbol{X},$$

where M^v is a 3×3 nonsingular matrix and $\simeq$ implies the equality sign up to a scale factor. We remark that in this formulation, all the information of camera parameters is included in M^v; we need not assume that the camera is calibrated. In this paper, we call M^v a *pointwise projection*.

3. Conic-based relationship between a conic in 3-D and its image

3.1. Conicwise projection

Since a conic in 3-D lies in a plane, embedding both the plane involving the conic and the image plane into $\mathcal{P}^2$ allows us to express a conic in 3-D and its images in the same form of

$$\boldsymbol{x}^{\mathrm{T}} C \boldsymbol{x} = 0, \qquad (3.1)$$

where C is the 3×3 symmetric matrix given by

$$C \simeq \begin{pmatrix} a & f & e \\ f & b & d \\ e & d & c \end{pmatrix}.$$

C is called a conic matrix. Note that we need not assume any calibration in this formulation. It follows from (3.1) that conics have a one-to-one correspondence with Cs i.e., 3×3 symmetric matrices up to scale. This is because one conic corresponds to one C up to scale and another conic corresponds to another C up to scale. C has six independent entries and only the ratios between them are significant. In other words, C bijectively corresponds to a point in $\mathcal{P}^5$. Hereafter, for a conic (3.1), we refer to $\boldsymbol{q} = (a, b, c, d, e, f)^{\mathrm{T}}$ as the *coordinates* of the conic.

Conics are transformed into conics under projective transformations. It is not difficult to see the relationship between a conic in 3-D (with conic coordinates $\boldsymbol{Q}$) and its image (with conic coordinates $\boldsymbol{q}^v$) observed from viewpoint v:

$$\boldsymbol{q}^v \simeq S^v \boldsymbol{Q}, \qquad (3.2)$$

where S^v is a nonsingular 6×6 matrix (see Fig. 1). We call S^v a *conicwise projection*.

3.2. Link between conicwise projection and pointwise projection

When points are subject to a pointwise projection M^v, its corresponding conic matrix C is transformed to

$$C^v \simeq (M^v)^{\mathrm{T}} C M^v. \qquad (3.3)$$

This equation indicates that each entry of C^v is expressed as the linear combination of the entries of C and that the coefficients are quadratic homogeneous in the entries of M^v. Each entry of S^v is thus quadratic homogeneous in the entries of M^v. This relationship is explicitly obtained in the following way.

If we define[1]

$$Q^v_{ijk\ell} \quad := \quad M^v_{ki}\, M^v_{\ell j} \quad (i,j,k,\ell \in \{1,2,3\})$$

and then rewrite (3.3) in terms of the entries of the conic coordinates, it follows that

$$C^v_{(i\,j)} \quad \simeq \quad \sum_{(k\,\ell)} \sigma(k,\ell)\, Q^v_{(i\,j)(k\,\ell)}\, C_{(k\,\ell)}, \qquad (3.4)$$

where $C_{(k\,\ell)}$ denotes the entry of the conic coordinates that is deduced from $C_{k\ell}$ ($C^v_{(i\,j)}$ is defined in the same manner). The notation $(\cdot\ \cdot)$ implies the symmetrization of the indices inside parentheses; we define that the symmetrized indices are aligned in such a way as $(1\,1),(2\,2),(3\,3),(2\,3),(3\,1),(1\,2)$. Moreover, function σ is defined by

$$\sigma(k,\ell) \quad := \quad \begin{cases} 1 & (k=\ell) \\ 2 & (k \neq \ell). \end{cases}$$

To be concrete, $C_{(k\ell)}$ and $Q^v_{(ij)(k\ell)}$ are given as follows:

$$\begin{aligned} C_{(k\ell)} &:= \frac{C_{k\ell}+C_{\ell k}}{2} \\ &= C_{k\ell} \qquad\qquad (\text{since } C^{\mathrm{T}} = C), \\ Q^v_{(ij)(k\ell)} &:= \frac{Q^v_{ijk\ell}+Q^v_{jik\ell}+Q^v_{ij\ell k}+Q^v_{ji\ell k}}{4} \\ &= \frac{M^v_{ki}M^v_{\ell j}+M^v_{kj}M^v_{\ell i}}{2}. \end{aligned}$$

Here we used tensor analysis. See [21] for details of tensor analysis and the notations.

From (3.2) and (3.4), we establish the link between a conicwise transformation S^v and its corresponding pointwise transformation M^v:

$$S^v_{(i\,j)(k\,\ell)} \quad \simeq \quad \sigma(k,\ell)\, M^v_{\underset{(i}{(k}}\, M^v_{\underset{j)}{\ell)}}\,. \qquad (3.5)$$

Note that, from the definition,

$$M^v_{\underset{(i}{(k}}\, M^v_{\underset{j)}{\ell)}} \quad = \quad \frac{M^v_{ki}M^v_{\ell j}+M^v_{kj}M^v_{\ell i}}{2}.$$

We see that we obtain S^v by taking the tensor product of M^v and itself, and then symmetrizing the rows and the columns respectively.

Based on (3.5), Sugimoto [26, 27] derived a linear algorithm for computing M^v (up to scale) from a given S^v.

[1] M_{ki} or $M_{\underset{i}{k}}$ denotes the (k,i) entry of a matrix M.

3.3. Algebraic properties of conicwise transformation

S^v is essentially determined by a pointwise projection M^v and, therefore, it has eight independent parameters. We thus see that S^v belongs to an eight-dimensional submanifold of all 6×6 matrices up to scale. In other words, if we are given an arbitrary 6×6 matrix up to scale, the matrix does not necessarily belong to the submanifold.

A 6×6 matrix up to scale has $36 - 1 = 35$ degrees of freedom and an eight-dimensional manifold has 8 degrees of freedom. To give the necessary and sufficient conditions on S^v for belonging to the submanifold constructed essentially by M^v, it is thus sufficient to derive $35 - 8 = 27$ algebraically independent constraints on entries of S^v. We derive here this necessary and sufficient conditions.

To obtain the constraints on the entries of S^v, we investigate (3.5) in detail. Without loss of generality, we may handle $\widehat{S^v}$ instead of S^v since $\widehat{S^v} = S^v D^{-1}$, where $D := \mathrm{diag}(1,1,1,2,2,2)$:

$$\widehat{S^v}_{(i\,j)(k\,\ell)} \quad := \quad \rho\, \frac{M^v_{ki}M^v_{\ell j}+M^v_{kj}M^v_{\ell i}}{2} \quad (\rho \neq 0).$$

We pay our attention to the particular cases where $i = j$ or $k = \ell$, from which it follows that

$$\begin{aligned} \widehat{S^v}_{(ii)(kk)} &= \rho (M^v_{ki})^2, \\ \widehat{S^v}_{(ii)(k\ell)} &= \rho M^v_{ki} M^v_{\ell i}, \\ \widehat{S^v}_{(ij)(kk)} &= \rho M^v_{ki} M^v_{kj}. \end{aligned}$$

It is now easy to see that the entries of $\widehat{S^v}$ satisfy the following relationship:

$$\begin{aligned} 2\widehat{S^v}_{(ij)(k\ell)} &= \frac{\widehat{S^v}_{(ii)(k\ell)} \cdot \widehat{S^v}_{(ij)(kk)}}{\widehat{S^v}_{(ii)(kk)}} + \frac{\widehat{S^v}_{(ii)(k\ell)} \cdot \widehat{S^v}_{(ij)(\ell\ell)}}{\widehat{S^v}_{(ii)(\ell\ell)}}. \\ 2\widehat{S^v}_{(ij)(k\ell)} &= \frac{\widehat{S^v}_{(jj)(k\ell)} \cdot \widehat{S^v}_{(ij)(kk)}}{\widehat{S^v}_{(jj)(kk)}} + \frac{\widehat{S^v}_{(jj)(k\ell)} \cdot \widehat{S^v}_{(ij)(\ell\ell)}}{\widehat{S^v}_{(jj)(\ell\ell)}}. \\ 2\widehat{S^v}_{(ij)(k\ell)} &= \frac{\widehat{S^v}_{(ii)(k\ell)} \cdot \widehat{S^v}_{(ij)(kk)}}{\widehat{S^v}_{(ii)(kk)}} + \frac{\widehat{S^v}_{(jj)(k\ell)} \cdot \widehat{S^v}_{(ij)(kk)}}{\widehat{S^v}_{(jj)(kk)}}. \\ 2\widehat{S^v}_{(ij)(k\ell)} &= \frac{\widehat{S^v}_{(ii)(k\ell)} \cdot \widehat{S^v}_{(ij)(\ell\ell)}}{\widehat{S^v}_{(ii)(\ell\ell)}} + \frac{\widehat{S^v}_{(jj)(k\ell)} \cdot \widehat{S^v}_{(ij)(\ell\ell)}}{\widehat{S^v}_{(jj)(\ell\ell)}}. \end{aligned}$$

When $i \neq j$ and $k \neq \ell$ (we have 9 cases for such i,j,k,ℓ), these equations make sense and indicate that each $\widehat{S^v}_{(ij)(k\ell)}$ is expressed in four different ways in terms of other entries of $\widehat{S^v}$. (The equations above may be rewritten as cubic constraints on the entries of $\widehat{S^v}$.) It is not difficult to see that any three of the four expressions are algebraically independent. (From any three expressions above, the other one is derived.) We thus have $9 \times 3 = 27$ algebraically independent (cubic) constraints on the entries of $\widehat{S^v}$, which are the necessary and sufficient constraints on the entries of $\widehat{S^v}$ for belonging to the eight-dimensional submanifold that is essentially constructed by M^v. In fact, 6×6 matrices up to scale satisfying these 27 constraints have a one-to-one correspondence with conicwise projections.

4. Multilinear forms

We consider a conic in 3-D with conic coordinates $\boldsymbol{Q}$, and its N images ($N \geq 2$). We obtain from (3.2)

$$\lambda^i \boldsymbol{q}^i \quad = \quad S^i \boldsymbol{Q} \qquad (i = 1, 2, \ldots, N),$$

where $\boldsymbol{q}^i$ denotes the conic coordinates of the image conic observed from viewpoint i and S^i denotes the conicwise projection onto the image i. Along with [10, 12], the above equations can be written as

$$S\boldsymbol{\xi} \quad = \quad \boldsymbol{0}, \tag{4.1}$$

where

$$S := \begin{bmatrix} S^1 & \boldsymbol{q}^1 & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ S^2 & \boldsymbol{0} & \boldsymbol{q}^2 & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S^N & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{q}^N \end{bmatrix}, \quad \boldsymbol{\xi} := \begin{bmatrix} \boldsymbol{Q} \\ -\lambda^1 \\ -\lambda^2 \\ \vdots \\ -\lambda^N \end{bmatrix}.$$

Since S in (4.1) has a nontrivial nullspace, we have

$$\text{rank} S \quad \leq \quad (6 + N) - 1 = N + 5.$$

This is equivalent to that all $(N + 6) \times (N + 6)$ submatrices of S have vanishing determinants. We see that any of these minors contains all the columns of S and that the coordinates in different images do not appear in the same column, from which it follows that all the determinants of $(N+6) \times (N+6)$ submatrices of S are multihomogeneous of degree $(1, 1, \ldots, 1)$ that is of the same degree in every N-plet of image coordinates. Moreover, it can be seen from the structure of S that any term of the determinants is factorized as a product of image coordinates multiplied by an expression involving only rows taken from at most six different conicwise projections. This observation means that this type of constraints exist only for at most six corresponding image conics.

S has the $(N+5)\{6N + (6+N)\} - (N+5)^2 = (N+5)(6N+1)$ degrees of freedom since S is a $6N \times (6+N)$ matrix whose rank is at most $(N+5)$. We thus have $6N(6+N) - (N+5)(6N+1) = 5(N-1)$ algebraically independent constrains on the entries of S.

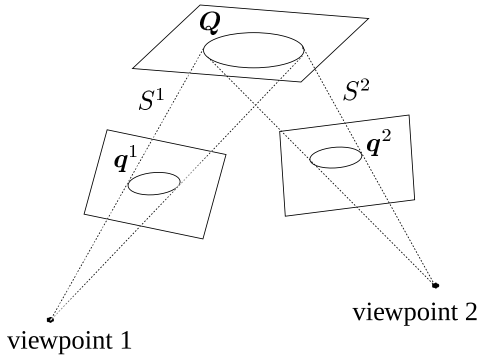

Figure 2. Two views of a conic in 3-D

5. Bilinear constraints

We here focus on the case of $N = 2$, that is, we assume that two corresponding image conics are given.

When we observe a conic in 3-D (with conic coordinates $\boldsymbol{Q}$) from two different viewpoints 1 and 2 by uncalibrated cameras (Fig. 2), we have

$$\begin{bmatrix} S^1 & \boldsymbol{q}^1 & \boldsymbol{0} \\ S^2 & \boldsymbol{0} & \boldsymbol{q}^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{Q} \\ -\lambda^1 \\ -\lambda^2 \end{bmatrix} \quad = \quad \boldsymbol{0}. \tag{5.1}$$

Since the 12×8 coefficient matrix in (5.1) is at most of rank 7, all the (8×8)-minors of the matrix have vanishing determinants. In taking the (8×8)-minors, we have ${}_{12}\mathrm{C}_8$ cases, only 5 of which are algebraically independent.

We can show that the five algebraically independent minors exist that involve the first four rows from one conicwise projection, the first three rows from the other conicwise projection—these seven rows are fixed—and any one row from the other five (two from the first projection and three from the second). (A brief sketch of the proof is given in the appendix.) An unfixed one row selection results in the five algebraically independent minors. For example, we can involve the minor obtained by the first five rows from S^1 and the first three rows from S^2:

$$\det \begin{bmatrix} S^1_{1\cdot} & q^1_1 & 0 \\ S^1_{2\cdot} & q^1_2 & 0 \\ S^1_{3\cdot} & q^1_3 & 0 \\ S^1_{4\cdot} & q^1_4 & 0 \\ S^1_{5\cdot} & q^1_5 & 0 \\ S^2_{1\cdot} & 0 & q^2_1 \\ S^2_{2\cdot} & 0 & q^2_2 \\ S^2_{3\cdot} & 0 & q^2_3 \end{bmatrix} \quad = \quad 0,$$

where $S^i_{j\cdot}$ and q^i_j denote the jth row of S^i and jth entry of $\boldsymbol{q}^i$, respectively. Laplace expansions of the minors allow us to express the five determinants in the bilinear forms in the coordinates of the two image conics. This indicates that the corresponding image conics satisfy the five independent bilinear constraints, and they have the form of

$$\sum_{i,j} {}_{IJ}B_{i,j} \cdot q_i^I q_j^J \quad = \quad 0,$$

where q_i^I is the ith entry of $\boldsymbol{q}^I$ and ${}_{IJ}B_{i,j}$ is defined as the determinant of a 6×6 matrix whose rows are taken only from rows of the two conicwise projections S^I and S^J. For instance, ${}_{IJ}B_{51} = \det[S^I_{1\cdot}\ S^I_{2\cdot}\ S^I_{3\cdot}\ S^I_{4\cdot}\ S^J_{2\cdot}\ S^J_{3\cdot}]$.

We see that we have 30 ${}_{IJ}B_{i,j}$'s that actually appear in the bilinear constraints. Since each conic correspondence gives five independent linear constraints on ${}_{IJ}B_{i,j}$'s, we can linearly estimate ${}_{IJ}B_{i,j}$'s up to a scale factor from six coplanar conics even if nothing is known about ${}_{IJ}B_{i,j}$. In fact, we face the case in robotic applications [2] where we have to deal with more than six coplanar conics, no pair of which is intersecting and nothing is known about them. We remark that knowledge about ${}_{IJ}B_{i,j}$, of course, reduces the number of required conics for estimating ${}_{IJ}B_{i,j}$.

Remark 5.1 Fixing another set of seven rows from the coefficient matrix in (5.1) yields also ${}_{IJ}B_{i,j}$'s, however, our selection of the seven fixed rows minimizes the number of ${}_{IJ}B_{i,j}$'s that actually appear in the bilinear constraints. Originally we have 36 ${}_{IJ}B_{i,j}$'s, six of which, i.e., ${}_{IJ}B_{k\ell}$ $(k = 4, 5, 6; \ell = 5, 6)$, do not appear in the above row set selection (cf. the appendix for this verification). □

Remark 5.2 ${}_{IJ}B_{i,j}$'s have a relationship with the homography between two images I and J. This is because conics are coplanar objects and two images of coplanar objects are related by the homography. The results of Sugimoto [26, 27] implicitly characterize this relationship. In this paper, however, we do not get into this direction. Detailed research results in this direction will be reported in another paper. □

6. Trilinear constraints

When we observe a conics in 3-D (with conic coordinates $\boldsymbol{Q}$) from three different viewpoints 1, 2 and 3 by uncalibrated cameras (Fig.3), we have

$$\begin{bmatrix} S^1 & \boldsymbol{q}^1 & \boldsymbol{0} & \boldsymbol{0} \\ S^2 & \boldsymbol{0} & \boldsymbol{q}^2 & \boldsymbol{0} \\ S^3 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{q}^3 \end{bmatrix} \begin{bmatrix} \boldsymbol{Q} \\ -\lambda^1 \\ -\lambda^2 \\ -\lambda^3 \end{bmatrix} = \boldsymbol{0}. \qquad (6.1)$$

All the (9×9)-minors of the coefficient matrix in (6.1) have vanishing determinants and only 10 of them are algebraically independent, which yields 10 algebraically independent trilinear constrains on the coordinates of three corresponding image conics.

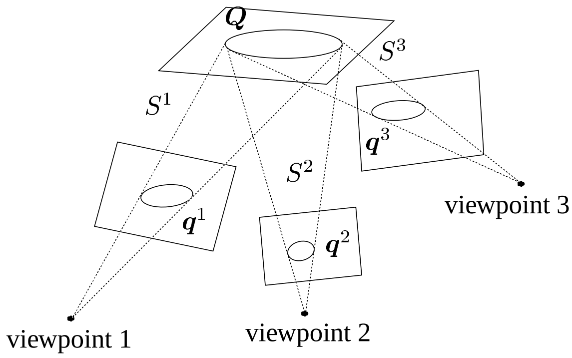

Figure 3. Three views of a conic in 3-D

Since the coefficient matrix contains one block with six rows for each image, we have two types of the (9×9)-minors: the minors whose rows are all from only two images and the minors whose rows are exactly from the three images.

As seen in the previous section, the determinants of the first type yield the bilinear constraints on the conic coordinates of the three pairs of the three image conics.

The determinants of the second type, on the other hand, yield the trilinear constrains on the coordinates of three image conics:

$$\sum_{i,j,k} {}_{IJK}T_{i,j,k} \cdot q_i^I q_j^J q_k^K \quad = \quad 0,$$

where ${}_{IJK}T_{i,j,k}$ is defined as the determinant of a 6×6 matrix whose rows are taken only from rows of the conicwise projections S^I, S^J and S^K. For example, ${}_{IJK}T_{121} = \det[S^I_{2\cdot}\ S^I_{3\cdot}\ S^I_{4\cdot}\ S^I_{5\cdot}\ S^I_{6\cdot}\ S^J_{1\cdot}]$. To compute 10 independent minors, we fix eight rows of the coefficient matrix in (6.1), i.e., six from one image, one from another image and one from the other image, and then we select one row from the other rows. This allows only 66 ${}_{IJK}T_{i,j,k}$'s to actually appear in the trilinear constraints[2] . Since each conic correspondence over three images gives 10 independent linear constraints on ${}_{IJK}T_{i,j,k}$'s, we can linearly estimate ${}_{IJK}T_{i,j,k}$'s up to a scale factor from seven coplanar conics even if nothing is known about ${}_{IJK}T_{i,j,k}$ nor ${}_{IJ}B_{i,j}$.

Remark 6.1 The number of algebraically independent constraints on $\boldsymbol{q}^1, \boldsymbol{q}^2$ and $\boldsymbol{q}^3$ derived from (6.1) is 10. In the previous section, we have already identified five algebraically independent constrains for the two-image case. This implies that we have 5 independent constraints between images 1 and 2, and other 5 independent constraints between images 2 and 3. The trilinear constraints on three corresponding image conics may, therefore, not be algebraically independent of the bilinear constraints on the conics between any pair of the three images. Clarifying the

[2] Originally, we have $6^3 = 216$ ${}_{IJK}T_{i,j,k}$'s.

relationship between bilinear constraints and trilinear ones is an open problem. □

7. Conclusion

We presented a study, based on conic correspondences, on the relationship between multiple images acquired by uncalibrated cameras. We employed the representation where a conic is represented as a point in the five-dimensional projective space. This representation allows us to handle image conics in the same way as image points.

A conicwise projection from 3-D to the image plane is represented as a 6×6 matrix (up to scale) belonging to the eight-dimensional submanifold determined by the corresponding pointwise projection. We gave the necessary and sufficient conditions on a conicwise projection for belonging to the submanifold determined by its corresponding pointwise projection.

We showed that the coordinates of corresponding image conics satisfy the multilinear constraints. In particular, the coordinates of two corresponding image conics satisfy five algebraically independent bilinear constraints. When a third image comes in, the coordinates of three corresponding image conics satisfy the trilinear constraints. These results are naturally extended to the case where more than three images are available: quadrilinear constraints for four image conics, pentalinear constraints for five image conics and hexalinear constraints for six image conics. We also showed that these types of constraints exist only for at most six different images and that the number of the algebraically independent constraints is $5(N-1)$, where N is the number of images.

Clarifying the relationship between ${}_{IJ}B_{i,j}$ in the bilinear constraints and ${}_{IJK}T_{i,j,k}$ in the trilinear constraints is left for future research.

Acknowledgments

This work was in part supported by Grant-in-Aid for Scientific Research of the Ministry of Education, Culture, Sports, Science and Technology of Japan under the contraction of 14380161.

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Appendix (Algebraically independent minors of a rectangular matrix)

We here consider minors of a rectangular matrix whose entries are all independent variables, and investigate algebraic independence among the minors. The Grassmann–Plücker identity below plays a significant role in this investigation.

Grassmann–Plücker identity Let A be a matrix with $|R| \leq |C|$, where $R = \mathrm{Row}(A)$ and $C = \mathrm{Col}(A)$. For $J, J' \subseteq C$ with $|J| = |J'| = |R|$ and $i \in J \backslash J'$, it holds that

$$\det A[R, J] \cdot \det A[R, J'] = \sum_{j \in J' \backslash J} \det A[R, J - i + j] \cdot \det A[R, J' + i - j],$$

where $A[R, J]$ means the submatrix of A having row set R and column set J, and $J - i + j$ is a short-hand notation for $(J \backslash \{i\}) \bigcup \{j\}$ in which the column j is put at the position of column i in J; similarly for $J' + i - j = (J' \bigcup \{i\}) \backslash \{j\}$. □

Let A be a $p \times n$ matrix with $p < n$ and $\mathrm{rank}\, A = r$ ($r \leq p$). Then, the number of $((r+1) \times (r+1))$-minors of A is ${}_p\mathrm{C}_{r+1} \times {}_n\mathrm{C}_{r+1}$. Since $\mathrm{rank}\, A = r$, A has $(p-r)(n-r)$ constraints on the entries of A, which indicates that we have $(p-r)(n-r)$ algebraically independent $((r+1) \times (r+1))$-minors of A.

In the two-view case, since

$$A^{\mathrm{T}} = \begin{bmatrix} S^1 & \boldsymbol{q}^1 & \mathbf{0} \\ S^2 & \mathbf{0} & \boldsymbol{q}^2 \end{bmatrix},$$

$p = 8$, $n = 12$, and $r = 7$. We thus have 5 algebraically independent (8×8)-minors of A.

We assume that $r = p - 1$ for simplicity. This assumption does not cause any essential difference in the subsequent result. Since $r = p - 1$, we can talk of $(p \times p)$-minors of A, and we have $(n - p + 1)$ algebraically independent minors. Letting

$$J^* := \{1, 2, 3, \ldots, p\},$$

and $\bar{J}^* = \mathrm{Col}(A) \backslash J^*$ $(= \{p+1, p+2, \ldots, n\})$, we define

$$J_\mu := J^* - 1 + \mu \qquad (\mu \in \bar{J}^*),$$

We now have the following theorem on independence of $(p \times p)$-minors of A.

Theorem Let $R = \mathrm{Row}(A)$. Then, $(n - p + 1)$ minors $\det A[R, J^*], \det A[R, J_\mu]$ $(\mu \in \bar{J}^*)$ are algebraically independent $(p \times p)$-minors of A. □

Proof. Instead of showing independence between $\det A[R, J^*]$ and $\det A[R, J_\mu]$ $(\mu \in \bar{J}^*)$ straightforwards, we will take an indirect proof. Since we know that $(n-p+1)$ algebraically independent minors exist, we show the other minors algebraically depend on $\det A[R, J^*]$ and $\det A[R, J_\mu]$ $(\mu \in \bar{J}^*)$.

To minor $\det A[R, J]$ $(J \subseteq \mathrm{Col}(A))$, we introduce distance d_J from $\det A[R, J^*]$:

$$d_J \quad := \quad |J^* \backslash J|.$$

We remark that $d_{J^*} = 0$ and $d_{J_\mu} = 1$ for $\mu \in \bar{J}^*$.

We classify all the $(p \times p)$-minors of A into $(n-p)$ groups depending on the distance from $\det A[R, J^*]$. Note that the maximum distance from $\det A[R, J^*]$ is $(n-p)$.

Let $\det A[R, J]$ be a minor with $d_J = 2$. Putting $J^* \backslash J = \{i_1, i_2\}$ and $J \backslash J^* = \{j_1, j_2\}$, where $i_1, i_2 \in J^*$ and $j_1, j_2 \in \bar{J}^*$, it follows from the Grassmann–Plücker identity that

$$\begin{aligned} &\det A[R, J^*] \cdot \det A[R, J] = \\ &\quad \det A[R, J^* - i_1 + j_1] \cdot \det A[R, J + i_1 - j_1] \\ &\quad + \det A[R, J^* - i_1 + j_2] \cdot \det A[R, J + i_1 - j_2]. \end{aligned}$$

We see that any minor that appears in the right-hand side of the above equation has the distance of 1 from $\det A[R, J^*]$. This indicates that $\det A[R, J]$ algebraically depends on $\det A[R, J^*]$ and the minors with distance 1 from $\det A[R, J^*]$. The important remark here is that applying the Grassmann–Plücker identity to a minor with distance 2 together with $\det A[R, J^*]$ reduces the distance by 1.

In the similar way, to any minor whose distance is greater than 2, we recursively apply the Grassmann–Plücker identity to see that it algebraically depends on $\det A[R, J^*]$ and the minors with distance 1.

To end the proof we have now only to show that the minors with distance 1 other than $\det A[R, J_\mu]$ $(\mu \in \bar{J}^*)$ algebraically depends on $\det A[R, J_\mu]$ $(\mu \in \bar{J}^*)$ and $\det A[R, J^*]$.

Let $J_{k\mu} := J^* - k + \mu$, where $k \in J^* \backslash \{1\}$ and $\mu \in \bar{J}^*$. Note that $d_{J_{k\mu}} = 1$.

We construct $(p+1) \times (p+1)$ matrix B whose principal $(p \times p)$-matrix is identical with $A[R, J^*]$ and whose $(p+1)$th column vector is identical with the vector concatenating $A[R, \{\mu\}]$ and 1. Moreover, the rest entries of B are assigned so that B degenerates. Taking the determinant of B, we have an algebraic equation with respect to $\det A[R, J_{k\mu}]$, $\det A[R, J_\mu]$ and $\det A[R, J^*]$.

We construct another $(p+1) \times (p+1)$ matrix C. We have two differences in the construction of C from B. One is that the principal $(p \times p)$-matrix of C is identical with $A[R, J_{k\mu}]$. The other is that the $(p+1)$th column vector of C is identical with the vector concatenating $A[R, \{\mu'\}]$ and 1, where $\mu' \in \bar{J}^* \backslash \{\mu\}$. Taking the determinant of C, we have another algebraic equation with respect to $\det A[R, J_{k\mu}]$, $\det A[R, J_\mu]$ and $\det A[R, J^*]$. Note that here we again employ the Grassmann–Plücker identity to reduce the distance from $\det A[R, J^*]$. Changing μ' leads to different algebraic equations.

From the obtained algebraic equations above, it is not difficult to see that the minors with distance 1 other than $\det A[R, J_\mu]$ $(\mu \in \bar{J}^*)$ algebraically depends on $\det A[R, J_\mu]$ $(\mu \in \bar{J}^*)$ and $\det A[R, J^*]$. $\square$

When we return to the case of two views, A has a special structure, i.e., the entries of the second half of the 7th row vector and the first half of the 8th row vector are all zero. This structure reduces the number of different terms that appear in the algebraically independent minors more than the case without any structure.

To determine the column-set order of our A that corresponds to J^*, we have essentially three cases. One is the case where 6 columns are from S^1 and 2 from S^2. Another case is that 5 columns are from S^1 and 3 from S^2. The third one is the case where 4 columns are from S^1 and 4 from S^2. From the structure of our A, it is easy to verify that the third case minimizes the number of different terms that actually appear in the algebraically independent minors. In fact, 36 different terms appear for the first case and 32 for the second one while for the third case only 30 different terms appear.