Optimal consensus set for digital line fitting

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Abstract. This paper presents a new method for fitting a digital line to a given set of points in a 2D image in the presence of noise by maximizing the number of inliers, namely the consensus set. By using a digital line model instead of a continuous one, we show that we can generate all possible consensus sets for digital line fitting. We present a deterministic algorithm that efficiently searches the optimal solution with the time complexity $O(N^2 \log N)$ and the space complexity $O(N)$ where N is the number of points.

Key words: digital geometry, optimization, consensus set, outliers

1 Introduction

Line fitting is an essential task in the field of image analysis. For instance, this procedure is useful for recognition [1], shape approximation [2] as well as parameter fitting [3]. There exist several optimal methods such as least-square fitting, least-absolute-value fitting or least median of squares (LMS) [3–5]. In these models, a continuous line model is used, defined by

$$\mathbf{L} = \{(x, y) \in \mathbb{R}^2 \ : \ ax + y + b = 0\}, \tag{1}$$

where $a, b \in \mathbb{R}$. The fitting is carried out through optimizing different cost functions. For instance, least-squares minimizes the sum of the geometric distances from all given points to the model. The solution can be obtained analytically, however it is not robust to the presence of outliers. Least-absolute values uses the vertical distances, instead of the geometric distances, for its minimization. Some efficient iterative algorithms have been proposed in the literature. However, if there are outliers, the solution is known to be unstable. In contrast, LMS minimizes the median of the vertical/geometric distances of all given points to the model. Thus, the fitting is robust as long as fewer than half of the given points are outliers [6].

In this paper, we present another globally optimal method that, given an arbitrary cloud of points, finds the line that minimizes the number of outliers, or alternatively maximizing the number of inliers, also called the consensus set. The idea of using such consensus sets were proposed for the RANdom SAmple Consensus (RANSAC) method [7], which is one of the most widely used in the field of computer vision. However RANSAC (and its variations) is inherently probabilistic in its approach, and do not guarantee any optimality while our method is both deterministic and optimal in the size of the consensus set.

In order to guarantee the optimality of consensus sets, we follow a digital geometry methodology [8] by using a digital line model instead of (1). This methodology is in fact natural given the assumption that our inputs are digital images. Besides, such a digital model allows us to distinguish between digitization-induced noise and actual noise in the case where the input data consist primarily of pixels. Related work using digital line models can be found in works such as digital line recognition [9, 10], and digital curve polygonalisation [2, 11] with and without the presence of noise. However, to the best of our knowledge, outliers, namely points which do not fit the model, have never been treated in the field of digital geometry.

The rest of the paper is as follows: in section 2 we expose the framework of our digital model. In section 3 we prove the optimality of our result by clarifying the relationship between our digital line model and its consensus sets. In section 4 we provide an algorithm for the computation of the fit, in terms of maximizing the number of inliers. We also show the computational time complexity of $O(N^2 \log N)$ and the space complexity of $O(N)$ with N the number of points. Section 5 provides a method for extracting the parameters from the fit. Section 6 is devoted to results and applications. Section 7 states some conclusions and perspectives.

2 The problem of digital line fitting

In this paper, we use the following digital line model in a discrete space $\mathbb{Z}^2$, where $\mathbb{Z}$ is the set of all integers. A digital line that is a digitization of $\mathbf{L}$ of (1), denoted by $\mathbf{D}(\mathbf{L})$. It is defined [15] by the set of discrete points simultaneously satisfying two inequalities:

$$\mathbf{D}(\mathbf{L}) = \{(x, y) \in \mathbb{Z}^2 \ : \ 0 \leq ax + y + b \leq w\} \tag{2}$$

where w is a given constant value. Geometrically, $\mathbf{D}(\mathbf{L})$ is a set of discrete points lying between two parallel lines $ax+y+b = 0$ and $ax+y+b = w$, and w specifies the vertical distance between these lines. For ideal images of digitized lines, from the discrete geometry viewpoint [8, 9], w should be $1 - 1/K$ where K is a very large constant if we expect that $\mathbf{D}(\mathbf{L})$ is 8-connected[5] and $-1 \leq a \leq 1$. In other words, this value of w is the minimum distance required to maintain the

[5] Such digital lines are specially called naive lines [8, 9].

(a) (b)

Fig. 1. A digital line that has one critical point $\boldsymbol{p}_1$ (a), and its rotated digital line that has a second critical point $\boldsymbol{p}_2$ (b).

connectivity of a digital line. For thicker contour images, w can be set to greater or equal to 1 [10, 11].

Concerning lines where $|a| > 1$, including vertical lines, we consider a digital line under a horizontal distance constraint, instead of a vertical one, between the two parallel lines. In that case, we simply exchange x and y in (2). Because the slope of a fitted digital line is not known in advance, we shall need to test both digital line models when fitting a digital line. Without loss or generality, we proceed in the remainder by using the vertical orientation in (2) for simplification.

Using this digital line model, our fitting problem is then described as follows: given a finite set of discrete points such that

$$\mathbf{S} = \{\boldsymbol{x} \in \mathbb{Z}^2 \ : \ i = 1, 2, \ldots, N\},$$

we would like to find a digital line $\mathbf{D}(\mathbf{L})$ such that $\mathbf{D}(\mathbf{L})$ contains the maximum number of points in $\mathbf{S}$. Points $\boldsymbol{x} \in \mathbf{S}$ are called inliers if $\boldsymbol{x} \in \mathbf{S} \cap \mathbf{D}(\mathbf{L})$; otherwise, they are called outliers. Note that in our problem, w is given as a constant and is fixed, so we do not need to find w when the fitting is accomplished.

3 Approach based on consensus sets

Our approach is focusing on inlier sets, also called consensus sets. Since the size of $\mathbf{S}$ is finite and each element $\boldsymbol{x} \in \mathbf{S}$ has finite coordinates, we easily notice that the number of different consensus sets for a digital line fitting of $\mathbf{S}$ is finite as well. Thus, if we can find all different consensus sets $\mathbf{C}$ from a given $\mathbf{S}$, we just need to check the size of each $\mathbf{C}$ and the one (or ones if there are ties) with maximal size as the optimal solution. The purpose of this section is to show that this is possible.

In the following, we give some notions related to digital lines. Two parallel lines that are given by the equations in (2) are called the support lines of a digital line. Discrete points that are on support lines are called critical points of a digital line. We then show the following proposition.

Proposition 1. *Let* $\mathbf{C}$ *be a consensus set of* $\mathbf{S}$ *for a digital line. It is possible to find a new digital line whose consensus set is the same as* $\mathbf{C}$ *such that it has at least two critical points.*

Fig. 2. A digital line that has no critical point (a), and its translated digital line that has one critical point $\boldsymbol{p}_1$ (b).

Proof. Let $\mathbf{D}$ be an initial digital line that contains all points in $\mathbf{C}$ as its inliers. Then, the following three cases can be considered when studying the critical points of $\mathbf{D}$.

1. Suppose that $\mathbf{D}$ has more than one critical points, then the proposition is already established in this case.
2. Suppose that $\mathbf{D}$ has one critical point $\boldsymbol{p}_1$ as illustrated in Figure 1 (a). In this case, we apply a rotation to $\mathbf{D}$ around $\boldsymbol{p}_1$ until finding another point $\boldsymbol{p}_2$ in $\mathbf{C}$ so that $\boldsymbol{p}_2$ becomes a critical point. The rotation is accomplished in such a way as to maintain the distance w between the support lines, and so that the support line on which there is not $\boldsymbol{p}_1$ is rotated around the point $\boldsymbol{p'}_1$ that is the projection of $\boldsymbol{p}_1$ on the line. Figure 1 (b) shows an example of a rotated digital line of Figure 1 (a) in order to find $\boldsymbol{p}_2$. Note that we can rotate $\mathbf{D}$ either clockwise or counterclockwise.
3. Suppose that $\mathbf{D}$ has no critical point as illustrated in Figure 2 (a). In this case, we first apply a translation to $\mathbf{D}$ in order to find a first critical point $\boldsymbol{p}_1$. Note that a translation can be made to any direction and the two support lines shall maintain the distance w between them. During such a translation, if more than one points are detected as critical points, then the proof is complete. If just one point $\boldsymbol{p}_1$ is detected, as illustrated in Figure 2 (b), then a rotation is made around $\boldsymbol{p}_1$ as mentioned in the previous case, in order to obtain a second critical point $\boldsymbol{p}_2$.

From this proposition, we see that we can find a digital line $\mathbf{D}(\mathbf{L})$ for any consensus set $\mathbf{C}$ of $\mathbf{S}$ such that it has at least two critical points. This is intuitively understandable, because when we move a digital line $\mathbf{D}(\mathbf{L})$ in the image plane, its consensus set $\mathbf{C}$ will be changed the moment a critical point goes out from $\mathbf{D}(\mathbf{L})$, namely, becomes an outlier, due to the motion of the line. Indeed, such a digital line $\mathbf{D}(\mathbf{L})$ can be constructed from a pair of points chosen from $\mathbf{S}$ such that they become critical points of $\mathbf{D}(\mathbf{L})$. Consequently, we can find all $\mathbf{C}$ from those $\mathbf{D}(\mathbf{L})$ constructed from pairs of points in $\mathbf{S}$. Note that the pair of points should not have the same x-coordinate with the vertical distance w, if we use (2) for computing a and b.

4 Algorithm

This section presents an algorithm that exhibits an optimal consensus set maximizing the number of inliers of a fitted digital line from a given set $\mathbf{S}$ of discrete points in 2D.

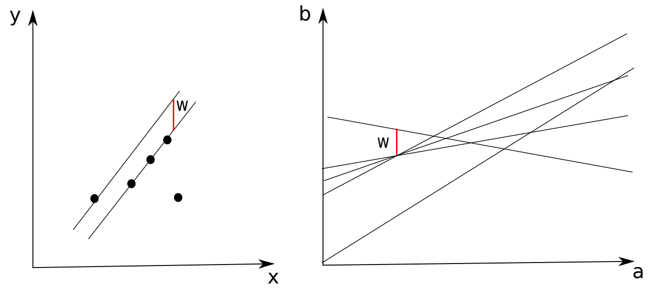

Fig. 3. A digital line of width w in the primal space (left) corresponds to a vertical line segment of length w in the dual space (right).

Our algorithm is inspired by the algorithm of LMS [12] working in the dual space of the following duality transform [13]; the dual space is also used for Hough Transform [16]: let $\boldsymbol{p} = (x_{\boldsymbol{p}}, y_{\boldsymbol{p}})$ be a 2D point in the primal space (x, y), then the dual of $\boldsymbol{p}$ is the line:

$$L^0_{\boldsymbol{p}} = \{(a, b) : x_{\boldsymbol{p}}a + b + y_{\boldsymbol{p}} = 0\}$$

in the dual space (a, b). Likewise, the dual of a non-vertical line $ax + y + b = 0$ in the primal space is the point (a, b) in the dual space.

Now, let us consider the dual-space interpretation of a digital line in the primal space, defined by (2). Because a digital line is regarded as a set of parallel lines whose slopes are $-a$ and y-intercepts are between $-b$ and $w - b$, it corresponds to a vertical line segment of length w, which is the distance between two parallel lines of a digital line, in the dual space as illustrated in Figure 3. Because points in $\mathbf{S}$ in the primal space are represented by lines in the dual space, the problem of finding the optimal consensus set in the primal set is equivalent to searching the best position of the vertical line segment of length w such that it intersects with as many lines as possible in the dual space, as illustrated in Figure 3.

Obviously, we cannot search everywhere in the dual space to find the best line segment. From Proposition 1, we know that, for any consensus set, there exists a digital line that has at least two critical points. Therefore, we first take one point $\boldsymbol{p} \in \mathbf{S}$, and consider it to be the first critical point of such a fitted digital line. Because $\boldsymbol{p}$ corresponds to a line $L^0_{\boldsymbol{p}}$ in the dual space, all digital lines for which $\boldsymbol{p}$ is a critical point correspond to the set of all the vertical line segments of length w having one of its endpoints on $L^0_{\boldsymbol{p}}$ in the dual space, as shown in Figure 4. The set of such digital lines, therefore, forms two strips in the dual space; one of them is bounded by $L_{\boldsymbol{p}}^0$ and $L_{\boldsymbol{p}}^1$, and another is bounded by $L_{\boldsymbol{p}}^0$ and $L_{\boldsymbol{p}}^2$, where

$$L_{\boldsymbol{p}}^1 = \{(a,b) : x_{\boldsymbol{p}}a + b + y_{\boldsymbol{p}} - w = 0\},$$
$$L_{\boldsymbol{p}}^2 = \{(a,b) : x_{\boldsymbol{p}}a + b + y_{\boldsymbol{p}} + w = 0\},$$

as illustrated in Figure 4. For simplification, we focus on the strip bounded by $L_{\boldsymbol{p}}^0$ and $L_{\boldsymbol{p}}^1$, because the following discussion is also valid for another strip bounded by $L_{\boldsymbol{p}}^0$ and $L_{\boldsymbol{p}}^2$.

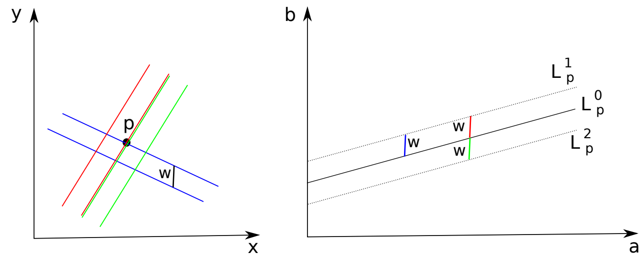

Fig. 4. Digital lines on which a point $\boldsymbol{p}$ is a critical point in the primal space (left), and those corresponding vertical line segments of length w in the dual space (right). In the dual space, a set of all such digital lines forms two strips, each of which is bounded by two lines $L_{\boldsymbol{p}}^0$ and $L_{\boldsymbol{p}}^i$ for $i = 1, 2$.

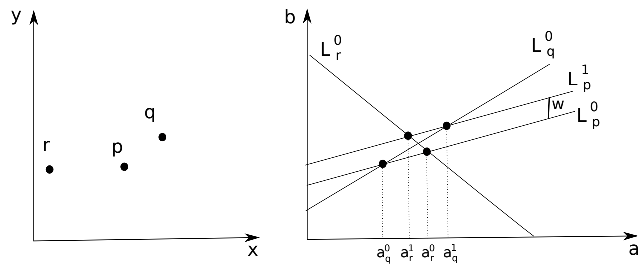

Fig. 5. Three points $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ in the primal space (left), and the corresponding lines $L_{\boldsymbol{p}}^0$, $L_{\boldsymbol{q}}^0$ and $L_{\boldsymbol{r}}^0$ in the dual space, with $a_{\boldsymbol{q}}^i$, $a_{\boldsymbol{r}}^i$ for $i = 0, 1$, the a-coordinates of the intersections of either $L_{\boldsymbol{q}}^0$ or $L_{\boldsymbol{r}}^0$ with $L_{\boldsymbol{p}}^i$ (right).

In order to know the number of inliers within every digital line on which $\boldsymbol{p}$ is a critical point in the primal space, we need to count the number of the other lines, $L_{\boldsymbol{q}}^0$ for $\boldsymbol{q} \in \mathbf{S}$ where $x_{\boldsymbol{q}} \neq x_{\boldsymbol{p}}$, that intersects with the strip at every moment of values a in the dual space. This is accomplished by checking the intersections of all the lines with the strip. Let us consider $\boldsymbol{q} = (x_{\boldsymbol{q}}, y_{\boldsymbol{q}})$ as one of such a point in $\mathbf{S}$ to be the second critical point of a fitted digital line. Geometrically, in the dual space, a digital line having two critical points $\boldsymbol{p}$ and $\boldsymbol{q}$ is given by the vertical line segment in the strip, one of whose endpoints is one of the intersections $(a^i_{\boldsymbol{q}}, b^i_{\boldsymbol{q}})$ of $L^i_{\boldsymbol{p}}$ and $L^0_{\boldsymbol{q}}$ for $i = 0, 1$. In fact, the digital lines corresponding to the vertical line segments between $a^0_{\boldsymbol{q}}$ and $a^1_{\boldsymbol{q}}$ in the strip always contain $\boldsymbol{q}$ as an inlier. For every $\boldsymbol{q} \in \mathbf{S}$ such that $x_{\boldsymbol{q}} \neq x_{\boldsymbol{p}}$, we thus calculate these two values $a^i_{\boldsymbol{q}}$, $i = 0, 1$, as illustrated in Figure 5.

Once $a^i_{\boldsymbol{q}}$, $i = 0, 1$ for every $\boldsymbol{q} \in \mathbf{S}$ are obtained, we sort all $a^i_{\boldsymbol{q}}$ in increasing order. For determining the moment of maximum intersections, a simple function $F(a)$ is used; after initially setting $F(a) = 0$ for every a, 1 is added when $L^0_{\boldsymbol{q}}$ enters the strip and -1 is added when $L^0_{\boldsymbol{q}}$ leaves the strip. The details for the calculation of $F(a)$ are shown in Algorithm 1; two values $f^i_{\boldsymbol{q}}$ for $i = 0, 1$ are used.

Algorithm 1 also considers another strip bounded by $L^0_{\boldsymbol{p}}$ and $L^2_{\boldsymbol{p}}$, instead of $L^0_{\boldsymbol{p}}$ and $L^1_{\boldsymbol{p}}$, as seen in Steps 4, 9 and 21. Note that depending on the strip, we calculate different b^c, as shown in Steps 21 and 22, because of the translation difference w between the two strips. We also note that the algorithm provides us with the set of parameter pair values (a^c, b^c) of all the fitted digital lines of (2) that give the optimal consensus sets.

The time complexity of the algorithm is $O(N^2 \log N)$, because we have N points in $\mathbf{S}$ and each $\boldsymbol{p} \in \mathbf{S}$ needs the complexity $O(N \log N)$ for sorting at most $2N - 2$ different values $a^i_{\boldsymbol{q}}$ for $\boldsymbol{q} \in \mathbf{S}$, $\boldsymbol{q} \neq \boldsymbol{p}$, and $i = 0, 1$. The space complexity is $O(N)$ because for each sorting we have at most $2N-2$ different pairs (a_k, f_k).

Because all inputs can be given as integers or rational numbers, all computations in Algorithm 1 can be made by using only rational numbers. This guarantees that all results obtained by Algorithm 1 contain no calculation error.

5 Feasible digital line parameters

Once we obtain an optimal consensus set $\mathbf{C}$ for digital line fitting to a given point set $\mathbf{S}$, we need the parameters of digital lines fitted to $\mathbf{C}$ for further applications. In general, the continuous line model such as (1) is used for estimating the parameters, for example, by applying the least squared method [3] to $\mathbf{C}$. However, we must be careful because this may change inliers. In such a case, a new $\mathbf{C}$ should be recalculated from a new estimated line, so that the iterative procedure may be necessary for renewing $\mathbf{C}$ with consecutive re-estimated line parameters.

In our case, however, since we use the digital line model such as (2) instead of (1), we do not need such an estimation method, and there is no possibility that parameter values obtained by $\mathbf{C}$ may produce a different $\mathbf{C}$. We can obtain all feasible solutions for the parameters of digital lines fitted to an obtained optimal $\mathbf{C}$ as follows: we just look for all feasible solutions (a, b) that satisfy the inequalities of (2) for all $(x, y) \in \mathbf{C}$.

Algorithm 1: Digital line fitting

input : A set $\mathbf{S}$ of N grid points, a value w
output: A list $\mathbf{V}$ of parameter value pairs (a^c, b^c) of the best fitted digital line

```
begin
    initialize Max = 0;
    foreach p ∈ S do
        for l = 1, 2 do
            initialize the array T[k] for k = 1, ..., 2N − 2;
            set j = 0;
            foreach q ∈ S such that x_q ≠ x_p do
                calculate a_q^i for i = 0, 1;
                if l = 2 then calculate a_q^2 and reset a_q^1 = a_q^2;
                if a_q^0 < a_q^1 then set f_q^0 = 1, f_q^1 = −1;
                else set f_q^0 = −1, f_q^1 = 1;
                set the pair (a_q^i, f_q^i), for i = 0, 1, in T[2j + i];
                j = j + 1;
            sort all the elements (a_k, f_k) for k = 1, ..., 2j in T with the values
            a_k as keys;
            initialize F = 1;
            for k = 1, ..., 2j do
                F = F + f_k;
                if F > Max then set Max = F, V = ∅;
                if F = Max then
                    set a^c = a_k;
                    if l = 1 then b^c = −a_k x_p − y_p;
                    else b^c = −a_k x_p − y_p + w;
                    put (a^c, b^c) in V;
    return V;
end
```

Such feasible solutions of digital lines are called preimages, and it is known that they have interesting properties [14]. For instance, a preimage of a digital line forms a convex polygon in the dual space that has at most four vertices.

6 Experiments

6.1 Practical complexity

While Section 4 showed that the theoretical time complexity for digital line fitting is $O(N^2 \log N)$, it is also important to show that the practical computation time corresponds to this theoretical result. For this purpose, we employed our fitting algorithm for several randomly generated sets of points of increasing size. Figure 6 shows the time variation with respect to the number of points N by using an $N^2 \log N$ regression. As the graph illustrates, the measured timings

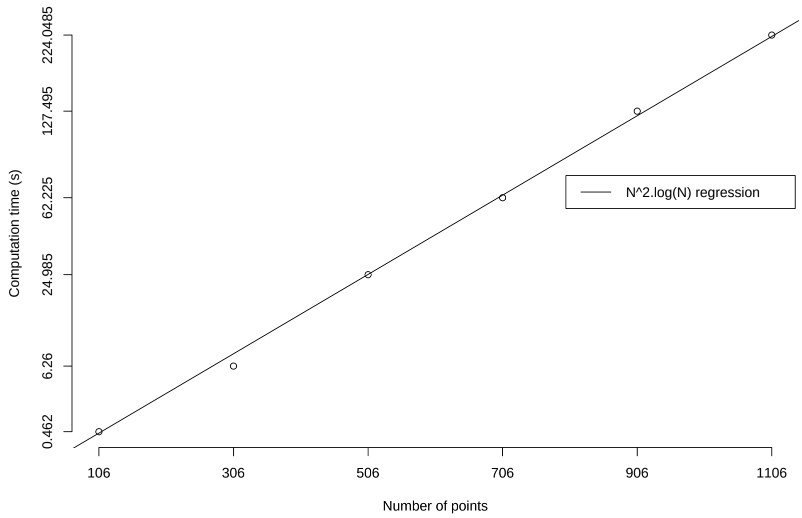

Fig. 6. Time versus number of points relations for digital line fitting.

follows the theoretical complexity. The algorithm was implemented in Matlab and not optimized for computational efficiency.

6.2 Ideal digitization image

We tested our method with an ideally digitized image. The image is made from two digital lines that are defined by a set of points $(x, y) \in \mathbb{Z}^2$ satisfying either $0 \leq x + 2y - 10 \leq 1$ or $0 \leq x - 2y \leq 1$, with some outliers. Our method is applied to fit a digital line to this set of points. The distance w is set to be one as same as the above given digital lines. Note that, in this case, we use the fixed-horizontal-distance model. Two optimal consensus sets are found using our method, as shown in Figure 7.

We compared our results with those of RANSAC. For comparison, the tolerance of RANSAC is set to 0.5; this value specifies the maximum distance of inliers from a fitted line. In this experiment, we use the continuous line model of (1) as in conventional RANSAC methods, and the horizontal distance as well as our method. Figure 8 shows the RANSAC results. We can see that RANSAC found only one consensus set, and besides it misses some inliers compared with the optimal consensus set in Figure 7. This is due to the fact that RANSAC is based on a random sampling, which provides no guarantee of optimality. However, the computation time is relatively rapid, thanks to its probabilistic strategy. Thus, in cases where it is sufficient to obtain an approximative solution for practical reasons, using RANSAC may be justified.

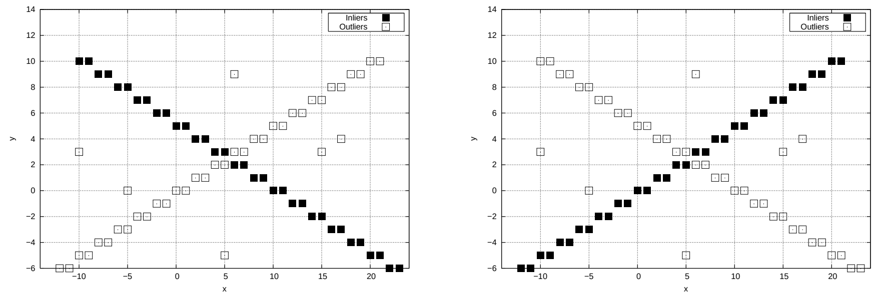

Fig. 7. Two optimal consensus sets obtained by our method for digital line fitting to an ideally digitized image.

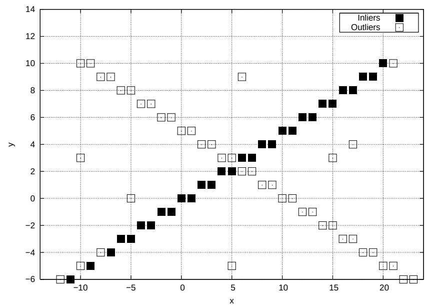

Fig. 8. A consensus set obtained by RANSAC for line fitting to an ideally digitized image.

6.3 Real image

We also tested our method with respect to a real image, as shown in Figure 9 (left), whose size is 520×693. Before applying our method, edge detection and mathematical morphological filtering are done for this image; the number of points in the image after this pre-processing is 5572 points. Our method is then applied in order to fit a digital line to the set of points. Figure 9 (right) shows the optimal consensus set, which includes 602 inliers, for digital line fitting. The distance w was set to be 1.

6.4 Polygonal contour images

We also tested polygonalisation using our method. It is tested using an iterative procedure by applying our method; after each iteration, we take the inliers off and apply our method to the remaining points.

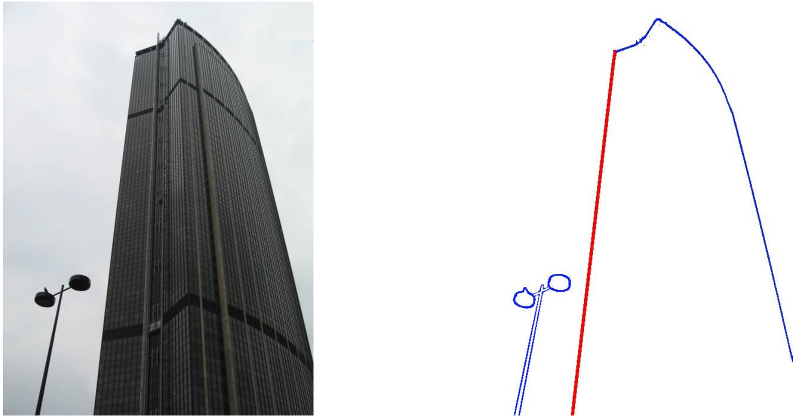

Fig. 9. An original image (left), and its optimal consensus set, in red color, of digital line fitting (right).

Figure 10 (left) shows the original polygonal contour image containing some noise whose size is 497×456. Figure 10 (right) shows the result after six iterations of applying our method for the polygonalisation. The consensus set obtained after each iteration is colored in red, blue, yellow, pink, cyan and green, respectively. The number of all points is 1960, and the sizes of the consensus sets are 297, 264, 186, 180, 119, and 104, respectively. The distance w is set to be 1.

Figure 11 (left) shows the contour image of a map of Mexico containing some noise whose size is 640×444. Figure 11 (right) shows that our method also works well on a complex non-concvexe shape. The consensus set obtained after each iteration is colored in red, blue, yellow, pink, cyan and green, respectively. The number of all points is 5324, and the sizes of the consensus sets are 262, 217, 196, 184, 177, and 161, respectively. Note that the distance w is set to be 3 for this example. Since our line fitting procedure does not take connectivity into account, it may be necessary to further decompose the fitted segment in a later procedure.

7 Conclusions

In this paper we have presented a new method for line fitting on discrete data – such as bitmap images, using a digital geometry (DG) approach. The DG approach has the advantage of clearly separating effects due to digitization on the one hand and noise on the other. Using our approach, we have proposed an optimal fitting method from the point of view of the maximal consensus set: we are guaranteed to fit digital lines with the least amount of outliers.

Our algorithm has a complexity that is identical to that of a parameter-free traditional line-fitting algorithm such as least median of squares regression [6], but allows us to define digital lines with precision, in the presence of outliers. Future work will include improving algorithmic complexity and more complete

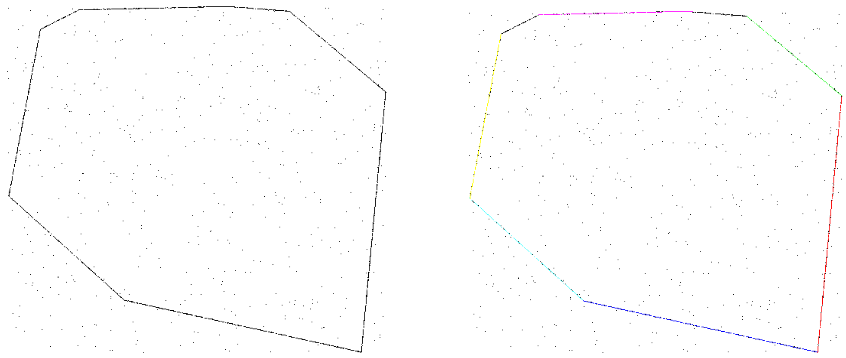

Fig. 10. A polygonal contour image with noise (left), and its result after six iterations of applying our method: the optimal consensus set obtained after each iteration is in red, blue, yellow, pink, cyan and green, respectively (right).

applications such as optimal polygonalisation by choosing a good value for the distance w automatically.

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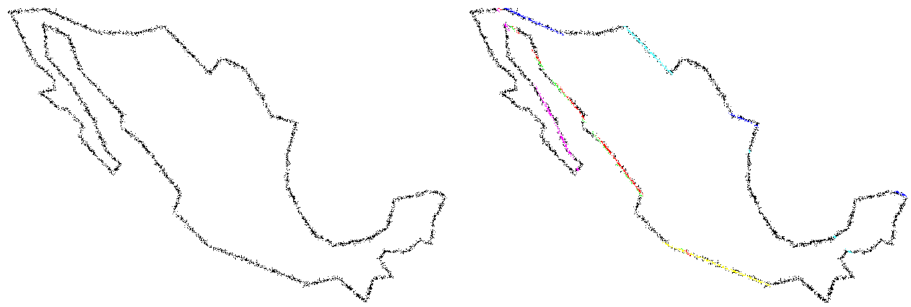

Fig. 11. A contour image of a map of Mexico with a noisy border (left), and its result after six iterations with our method: the optimal consensus set obtained after each iteration is in red, blue, yellow, pink, cyan and green, respectively (right).

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